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AN ABSTRACT NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION

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ABSTRACT

We study the nonlinear Volterra equation

$$(*)\quad \begin{cases} u'(t) + Bu(t) + \int_0^t a(t-s)Au(s)ds \ni F(t) & (0 < t < \infty) \quad (' = d/dt) \\ u(0) = u_0 , \end{cases}$$

as well as the corresponding problem with infinite delay

$$(**)\quad \begin{cases} u'(t) + Bu(t) + \int_{-\infty}^t a(t-s)Au(s)ds \ni f(t) & (0 < t < \infty) \\ u(t) = h(t) & (-\infty < t \le 0) . \end{cases}$$

Under various assumptions on the nonlinear operators A, B and on the given functions a, F, f, h existence theorems are obtained for (*) and (**), followed by results concerning boundedness and asymptotic behaviour of solutions on $(0 \le t < \infty)$; two applications of the theory to problems of nonlinear heat flow with "infinite memory" are also discussed.

AMS (MOS) Subject Classifications: 45D05, 45K05, 45N05, 45G99, 45M05, 35B40, 35K55, 35K60, 47G05, 47H05, 47H10, 47H15

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AN ABSTRACT NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION

M. G. Crandall*, S.-O. Londen**, and J. A. Nohel***

1. Introduction and Summary of Results

We study the nonlinear Volterra equation

$$(1.1)\quad \begin{cases} u'(t) + Bu(t) + \int_0^t a(t-s)Au(s)ds \ni F(t) & (0 < t < \infty) \quad (' = d/dt) \\ u(0) = u_0, & \end{cases}$$

as well as the corresponding problem with infinite delay

$$(1.1_\infty)\quad \begin{cases} u'(t) + Bu(t) + \int_{-\infty}^t a(t-s)Au(s)ds \ni f(t) & (0 < t < \infty) \\ u(t) = h(t) & (-\infty < t \le 0) . \end{cases}$$

Under various assumptions on the nonlinear operators A, B and on the given functions a, F, f, h existence theorems are obtained for (1.1) and (1.1_∞), followed by results concerning boundedness and asymptotic behaviour of solutions; two applications illustrating the theory to problems of heat flow "with memory" are also discussed. This work was partly motivated by Barbu [2]; see below.

The technical conditions appropriate to various circumstances are somewhat cumbersome and distracting to state. We therefore collect the assumptions common to most of our results under the name "general assumptions."

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<u>The General Assumptions.</u> Let H be a real Hilbert space and W a real reflexive Banach space satisfying

(1.2) $$W \subset H \subset W'$$

where W' is the dual of W. It is assumed that the injections in (1.2) are continuous and dense and $\langle w', w\rangle = (w', w)$ for $w' \in H$, $w \in W$ where $\langle w', w\rangle$ is the value of $w' \in W'$ at $w \in W$ and $(\ ,\ )$ is the inner product of H. We denote the norm in H by $|\cdot|$ and the norm in W by $\|\cdot\|$. Let $\psi : W \to (-\infty, \infty]$ and $\varphi : H \to (-\infty, \infty]$ be convex, lower semicontinuous (l.s.c.) and proper functions and define

(1.3) $$A = \partial\psi, \quad B = \partial\varphi,$$

where $\partial\psi$, $\partial\varphi$ are the subdifferentials of ψ and φ respectively (see, e.g., [5]). Then A and B are (possibly multivalued) maximal monotone operators from W and H to W' and H respectively. Define $\psi_H : H \to (-\infty, \infty]$ by

(1.4) $$\psi_H(u) = \lim_{r \downarrow 0} \inf\{\psi(v) : v \in W \text{ and } |v - u| < r\}.$$

ψ_H is automatically l.s.c. and ψ_H is convex since ψ is convex. ψ_H is the largest l.s.c. function on H satisfying $\psi_H \le \psi$ on W. We assume that

(1.5) $$\psi_H(u) = \psi(u) \text{ for } u \in W.$$

Let $A_H = \partial\psi_H$; A_H is maximal monotone in H and, in view of (1.5), has the property

$$(1.6) \qquad A_H u \subset Au \quad \text{for} \quad u \in W .$$

This follows from the implication: $u \in W$, $v \in H$ and $\psi_H(z) \geq \psi_H(u) + (v, z - u)$ for $z \in H \Rightarrow \psi(z) \geq \psi(u) + \langle v, z - u\rangle$ for $z \in W$ when (1.5) holds. Note that if $\tilde{\psi} : H \to (-\infty, \infty]$ defined by

$$\tilde{\psi}(u) = \begin{cases} \psi(u), & u \in W \\ +\infty, & u \in H\backslash W \end{cases}$$

is l.s.c., then $\tilde{\psi} = \psi_H$ and (1.5) holds. Moreover, $\tilde{\psi}$ is l.s.c. if $\lim\limits_{\|u\| \to \infty} \psi(u) = +\infty$.

The Yosida approximations A_λ of A_H can be defined for $\lambda > 0$ by

$$A_\lambda = \lambda^{-1}(I - J_\lambda), \; J_\lambda = (I + \lambda A_H)^{-1} ;$$

see [5] for the properties of A_λ. Relating A_λ and B we assume there exists $\beta \in [0, \infty)$ such that

$$(1.7) \qquad (w, A_\lambda u) \geq -\beta(|w|^2 + |u|^2 + 1) \quad \text{for } u \in H,\ w \in Bu,\ \lambda \in (0,1].$$

We will also require the compactness assumption

(1.8) For every $K > 0$, $\{u \in H : |\varphi(u)| + |u| \leq K\}$ is precompact in W. In particular, we assume $D(\varphi) \subseteq W$.

Finally, as regards the kernel a, we will require that the following conditions are satisfied.

<u>Conditions (a)</u>:

(1.9) $a(t)$ is locally absolutely continuous on $[0, \infty)$.

$$(1.10)\quad \begin{cases} \text{For every } T>0 \text{ there is a } K_T>0 \text{ such that} \\ v\in L^2(0,T;H),\ d_1,d_2\in[0,\infty) \text{ and} \\ \int_0^t (a*v(s),v(s))ds \le d_1 + d_2 \max_{0\le s\le t} \left|\int_0^s v(\tau)d\tau\right|,\quad 0\le t\le T, \\ (\text{where } a*v(t) = \int_0^t a(t-s)v(s)ds) \text{ imply} \\ \left|\int_0^t v(s)ds\right| \le K_T(\sqrt{d_1}+d_2),\quad 0\le t\le T, \\ \text{and } \left|\int_0^t (a*v(s),v(s))ds\right| \le K_T(d_1+d_2^2),\quad 0\le t\le T. \end{cases}$$

Note that if $v\in L^2(0,T_0;H)$, where $T_0<T$ satisfies the assumptions of (1.10) on $[0,T_0]$, then v extended as 0 on $(T_0,T]$ satisfies the same conditions on $[0,T]$. Thus, without loss of generality, $T\to K_T$ can be assumed nondecreasing. This concludes the general assumptions.

Some remarks on Conditions (a) are appropriate before proceeding to the statement of the main results. Conditions (a) abstract what is actually used in the proofs and are stated in this form for simplicity of presentation. Moreover, as stated these conditions are perfectly sensible for operator valued kernels and our results hold in this generality. A general sufficient condition which implies Conditions (a) is formulated in Theorem (a) of Appendix (a), and this is used in turn to verify:

<u>Proposition (a)</u>. <u>Let</u> a <u>satisfy either the conditions</u>

$$(a_1)\quad \begin{cases} a,a'\in L^1_{loc}([0,\infty);R)., \\ a(0)>0 \\ a' \ \underline{\text{is of bounded variation locally on}}\ [0,\infty), \end{cases}$$

or the conditions

$$(a_2) \quad \begin{cases} a(t) \text{ is nonnegative, decreasing and} \\ \text{convex on } [0,\infty),\ a(0) > 0 \text{ and} \\ a \in C^2((0,\infty)) \cap C([0,\infty)) . \end{cases}$$

Then a satisfies Conditions (a). Moreover, if $a = a_1 + a_2$ with a_i satisfying the Conditions (a_i), then a satisfies Conditions (a).

Thus a broad class of interesting kernels satisfy Conditions (a).

Proposition (a) is proved in Appendix (a).

Our first existence result is:

Theorem 1. Let the general assumptions (1.2)-(1.10) be satisfied. Further assume that $A = \partial\psi$ is single-valued and $D(A) = W$. Then for every $F \in W^{1,1}_{loc}([0,\infty); H)$ and $u_0 \in D(\varphi)$ equation (1.1) has a solution u in the sense

(i) $u \in C([0,\infty); W)$,

(ii) $u' \in L^2_{loc}([0,\infty); H)$,

(iii) $F - (u' + a * Au) \in L^2_{loc}([0,\infty); H)$,

(iv) $F(t) - u'(t) - a * Au(t) \in Bu(t)$ a.e. $t \geq 0$.

Moreover

(v) $\int_0^t Au(s)ds \in L^\infty_{loc}([0,\infty); H)$.

Theorem 1 is proved in Section 2.

Remarks: Since $A = \partial\psi : W \to W'$ is assumed to be single-valued and everywhere defined on W, A is continuous from the strong topology

of W into the weak topology of W'; see [17]. Thus by (i) above, $t \to Au(t)$ is continuous into the weak topology of W' and $a * Au$ is unambiguously defined with values in W'. Moreover, by (v) and $a * Au(t) =$

$$a(0) \int_0^t Au(s)ds + a' * (\int_0^t Au(s)ds), \quad a * Au \in L^\infty_{loc}(0,\infty; H);$$

the integrals are taken in the sense of Bochner.

The spaces W and W' enter in Theorem 1 as a technical device corresponding to the fact that we can obtain estimates in H of u and integrals of Au under the hypotheses of Theorem 1, but we cannot obtain estimates on Au in H. These estimates are obtained in Section 2 after preliminary results, of some independent interest, dealing with the regularized equation

$$(1.11) \qquad \begin{cases} u_\lambda'(t) + Bu_\lambda(t) + \varepsilon A_\lambda u_\lambda(t) + a * A_\lambda u_\lambda(t) \ni F(t) \\ \lambda, \varepsilon > 0, \ 0 \le t < \infty, \ u_\lambda(0) = u_0 . \end{cases}$$

After establishing existence and uniqueness of solutions of (1.11) for a fixed λ, $\varepsilon > 0$, a priori estimates are obtained which enable us to pass to the limit as $\lambda \to 0^+$ keeping $\varepsilon > 0$ fixed. Then using a priori estimates independent of $\varepsilon > 0$, Theorem 1 is proved on letting $\varepsilon \to 0^+$. The compactness assumption (1.8) concerning φ and properties of maximal monotone operators come into play in the passages to the limit as $\lambda \to 0^+$ and then as $\varepsilon \to 0^+$.

Under suitable assumptions estimates on Au in $L^2_{loc}([0,\infty); H)$ can be obtained. Then existence results can be proved in which neither A

nor B is required to be single-valued. For example, we have:

Theorem 2. *Let the general assumptions* (1.2)-(1.10) *be satisfied with* $W = H = W'$ (so $\psi_H = \psi$, $A_H = A$, etc.). *Assume, in addition that for each* $r > 0$ *there is a number* $k(r)$ *such that*

(1.12) $\quad k(r)(1 + |w|) \geq |v|$ *for* $v \in Au$, $w \in Bu$ *and* $|u| \leq r$.

Then for every $F \in W^{1,1}_{loc}([0,\infty); H)$ *and* $u_0 \in D(\psi) \cap D(\varphi)$ *equation* (1.1) *has a solution* u *satisfying* $u, u' \in L^2_{loc}([0,\infty); H)$, *and there exist* $v, w \in L^2_{loc}([0,\infty); H)$ *such that* $v(t) \in Au(t)$, $w(t) \in Bu(t)$ a.e. $(0 \leq t < \infty)$ *and* $u' + w + a * v = F$ a.e. $(0 \leq t < \infty)$.

Theorem 2 is proved together with Theorem 1 in Section 2.

The next task is to discuss the boundedness and asymptotic behaviour of solutions of equation (1.1). Two results of this type, motivated by analogous ones of interest in the stability theory of real scalar Volterra equations, are given. They seem typical of what one might expect to prove concerning solutions of (1.1) provided by results like Theorems 1 and 2.

Theorem 3 (i). *Let the general assumptions* (1.2)-(1.10) *be satisfied and* u, v, w *be given satisfying the conclusions of Theorem* 2. *Assume also that* $u(0) = u_0 \in D(\varphi) \cap D(\psi)$, $\beta = 0$ *in* (1.7), $\inf_{u \in W} \psi(u) > -\infty$, a *satisfies conditions* (a_2), *and there is a* $\delta \geq 0$ *for which*

(1.13) $\quad |F(t)| \leq \delta a(t)$, $|F'(t)| \leq -\delta a'(t)$ $(0 \leq t < \infty)$.

Then

$$(1.14)\quad \begin{cases} \text{(a)} \quad \sup\limits_{0 \le t < \infty} Q_a(v;t) < \infty \\ \text{(b)} \quad |a * v(t)|^2 \le 2a(0)Q_a(v;t) \quad (0 \le t < \infty) \\ \text{(c)} \quad \sup\limits_{t \ge 0} \psi(u(t)) < \infty, \end{cases}$$

where Q_a is defined by

$$Q_a(v;t) = \int_0^t (a * v(\tau), v(\tau))d\tau, \quad v \in L^2_{loc}([0,\infty); H) .$$

(ii) If also $\inf\limits_{u \in H} \varphi(u) > -\infty$ and $\lim\limits_{\|u\| \to \infty} \psi(u) = \infty$ then

$$(1.15)\quad \begin{cases} \text{(a)} \quad \sup\limits_{0 \le t < \infty} \|u(t)\| < \infty \\ \text{(b)} \quad \sup\limits_{0 \le t < \infty} |\varphi(u(t))| < \infty \\ \text{(c)} \quad u \in UC([0,\infty); W) \end{cases}$$

where $UC([0,\infty); W)$ is the set of uniformly continuous functions with values in W.

(iii) If in addition to the conditions above, $a'(t) \not\equiv 0$ and A satisfies the conditions of Theorem 1, then

$$(1.16)\qquad v(t) \to 0 \text{ weakly in } W' \text{ as } t \to \infty$$

and if $A^{-1}(0)$ is a singleton then

$$(1.17)\qquad u(t) \to A^{-1}(0) \text{ strongly in } W \text{ as } t \to \infty .$$

Theorem 3 is proved in Section 3.

Remarks. Theorem 3, as stated, does not apply directly to the solutions of (1.1) given by Theorem 1 even when the additional assumptions on ψ, φ, a, F

are satisfied since with $v = Au$, u as in Theorem 1, we do not have $v \in L^2_{loc}([0,\infty); H)$. The principal difficulty lies in that the defining expression for $Q_a(v;t)$ does not have a clear meaning for $v \in L^\infty_{loc}([0,\infty); W')$. However, for the solutions of (1.1) constructed in the proof of Theorem 1, this expression may be assigned a meaning and the results of Theorem 3 remain valid. This point is discussed at the end of the proof of Theorem 3 in Section 3.

Moreover, as will be clear from the proofs, if F is compactly supported and a is a kernel of positive type ([15]), rather than convex and nonincreasing, the conclusions of Theorem 3 remain valid.

In the next result the somewhat artificial condition (1.13) (see, however, Proposition 5.2 and the first example of Section 6) on F is replaced by $F \in L^2(0,\infty; H)$ and a need not be of positive type.

Theorem 4. Let the general assumptions (1.2)-(1.10) *be satisfied and* u, v, w *be given satisfying the conclusions of* Theorem 2 *where* $F \in L^2(0,\infty; H)$. *Assume also that* $u_0 \in D(\varphi) \cap D(\psi)$, $\inf_{u \in W} \psi(u) > -\infty$ *and there exist* α, $\delta > 0$

$$(1.18)\quad \begin{cases} \text{(i)}\ (w,v) \geq \alpha |v|^2 \ \underline{\text{for}}\ w \in Bu,\ v \in Au,\ u \in H \\ \underline{\text{and}} \\ \text{(ii)}\ \limsup_{T\to\infty}\ \inf_{-\infty<\sigma<\infty} \int_0^T \cos(\sigma t)a(t)dt \geq \delta - \alpha . \end{cases}$$

Then $\sup_{t>0} \psi(u(t)) < \infty$ *and* $v \in L^2(0,\infty; H)$.

The proof of Theorem 4 is given in Section 4.

The problem $(1.1)_\infty$ may be reduced, in the standard way, to a problem of the form (1.1) and the above results then applied. This is carried out in Section 5. Finally, in Section 6, we consider two examples to illustrate the theory.

Equation (1.1) has been studied by Barbu [2], [4] using energy functions [9]. Theorems 1 and 3 extend his main results in several directions. Barbu's existence theorem requires, in addition to the assumptions of Theorem 1, the kernel a to be positive, decreasing and convex, $\beta = 0$ in (1.7) and a number of restrictive technical conditions. Correspondingly, our proofs appear to us to be more illuminating, direct and complete. See the end of Section 2 concerning the generality afforded by allowing $\beta > 0$. Similar differences exist between our Theorem 3 and the version of [2]. Theorems 2 and 4 have no direct analogues in [2], and we have not stated an analogue of [2, Remark 3.1], which is not quite clear. See the amended version in [4]. However, from the proofs one can easily invent results of this type.

The special case of (1.1) in which $Au = Bu$ has been studied by MacCamy [14] by a different method essentially only under conditions a_2. When $Au = Bu$, (1.1) is formally equivalent to the integral equation

$$\text{(I)} \qquad u(t) + \int_0^t b(t-\tau)Au(\tau)d\tau \ni H(t), \quad 0 \leq t < \infty,$$

in which $b(0) = 1$, $b'(t) = a(t)$ and $H(t) = \int_0^t F(\tau)d\tau$.

Equation (I) has also recently been studied in Hilbert space by Barbu [3] and by S.-O. Londen [11]; existence, uniqueness and results for behaviour of solutions as $t \to \infty$ are obtained in [11] under more general assumptions than in [3]. Since the assumption (1.8) (or some similar compactness condition) is not made in [11], the results of [11] are also more general than those obtained in Theorems 2 and 4 in the special case $Au = Bu$. It should also be noted that uniqueness of solutions of (1.1) is not claimed in any of our principal results.

Let us also point out that the case $B \equiv 0$ in (1.1) is ruled out by the compactness assumption (1.8) (unless H is finited dimensional). When $B \equiv 0$ (1.1) is formally equivalent to the equation

$$\frac{d^2u}{dt^2} + a(0)Au(t) + \int_0^t a'(t - \tau)Au(\tau)d\tau \ni F'(t) .$$

Existence for this problem has recently been studied by Londen [12], [13]. The case when A is a linear second order partial differential operator has been analyzed by Dafermos [7], [8]. Also note that problems related to the ones considered here have been considered by Artola [1].

Finally, let us remark that this paper is an outgrowth of a seminar held in Madison, Wisconsin, during 1974-75. We acknowledge with pleasure the helpful discussions with colleagues and students, in particular with W. Rudin, D. F. Shea, Luc Tartar and O. Staffans.

2. Proof of Theorems 1 and 2

We begin with a general result (Lemma 2.1) and its consequence (Corollary 2.1) which will be applied to the regularized equation (1.11).

Consider the initial value problem

$$\frac{dw}{dt} + Bw \ni G(w);\ w(0) = w_0 . \tag{2.1}$$

Concerning (2.1) we prove:

Lemma 2.1. Let T_0 be given. Let

$$B \text{ be a maximal monotone graph in } H \times H , \tag{2.2}$$

$$G : C([0, T_0]; \overline{D(B)}) \to L^1(0, T_0; H) , \tag{2.3}$$

and let there exist a constant $M > 0$ such that

$$\|G(u) - G(v)\|_{L^\infty(0,t;H)} \le M\|u - v\|_{L^\infty(0,t;H)} \quad (0 \le t \le T_0) , \tag{2.4}$$

for $u, v \in C([0, T_0]; \overline{D(B)})$. If $w_0 \in \overline{D(B)}$, *then the initial value problem (2.1) has a unique solution w in the sense that w is a weak solution* (see [5; Def. 3.1]) *of the initial value problem*

$$\frac{dw}{dt} + Bw \ni E(t);\ w(0) = w_0 , \tag{2.5}$$

where $E(t) = G(w)(t)$; *in particular,*

$$w \in C([0, T_0]; H) \text{ and } w(t) \in \overline{D(B)} \text{ on } [0, T_0] . \tag{2.6}$$

If, in addition, $B = \partial\varphi$, *where* $\varphi : H \to (-\infty,\infty]$ *is convex, l.s.c. and proper, and if* $w_0 \in D(\varphi)$ *and* $G(w)(t) \in L^2(0, T_0; H)$, *then* w *is a strong solution* (see [5; Def. 3.1]) *of* (2.5) *and*

$$\frac{dw}{dt} \in L^2(0, T_0; H) . \tag{2.7}$$

Proof of Lemma 2.1. Consider the initial value problem

$$\frac{dw}{dt} + Bw \ni G(v); \; w(0) = w_0 \in \overline{D(B)} \tag{2.8}$$

where $v \in C([0, T_0]; \overline{D(B)})$ is given. By (2.3) $G(v)(t) \in L^1(0, T_0; H)$ and so from (2.2), $w_0 \in \overline{D(B)}$, and from [5; Theorem 3.4] it follows that (2.8) has a unique weak solution on $[0, T_0]$ which we denote by Tv; in particular, $Tv \in C([0, T_0]; \overline{D(B)})$. Furthermore, recalling [5; Lemma 3.1] and (2.4) yields

$$\left\{ \begin{aligned} &\|Tu - Tv\|_{L^\infty(0,t;H)} \le \int_0^t \|G(u) - G(v)\|_{L^\infty(0,s;H)} ds \\ &\le M \int_0^t \|u - v\|_{L^\infty(0,s;H)} ds, \qquad 0 \le t \le T_0, \end{aligned} \right. \tag{2.9}$$

for $u, v \in C([0, T_0]; \overline{D(B)})$. We claim that the mapping T has a unique fixed point. For, iterating (2.9) one obtains by a straightforward induction

$$\|T^n u - T^n v\|_{L^\infty(0,T_0;H)} \le \frac{M^n T_0^n}{n!} \|u - v\|_{L^\infty(0,T_0;H)}. \tag{2.10}$$

Thus, for n sufficiently large, T^n is a strict contraction on $C([0, T_0]; \overline{D(B)})$, and consequently, T has a unique fixed point $w \in C([0, T_0]; \overline{D(B)})$ which solves (2.1) as asserted in the first part of Lemma 2.1.

The second part of Lemma 2.1 follows immediately from [5; Th. 3.6].

Remark. The conclusions of Lemma 2.1 remain unchanged if the Lipschitz condition (2.4) is weakened to

$$\|G(u) - G(v)\|_{L^\infty(0,t;H)} \le \mu(t) \|u - v\|_{L^\infty(0,t;H)}, \qquad 0 \le t \le T_0 ,$$

for $u, v \in C([0,T_0]; \overline{D(B)})$ and where $\mu \in L^1(0,T_0)$. Moreover, the proof of the first part of Lemma 2.1 is valid without change if B is m-accretive in a Banach space X.

Lemma 2.1 will be applied to the regularized problem

$$(2.11) \qquad \begin{cases} u_\lambda' + Bu_\lambda(t) + \varepsilon A_\lambda u_\lambda(t) + a * A_\lambda u_\lambda(t) \ni F(t) \\ u_\lambda(0) = u_0 . \end{cases}$$

<u>Corollary 2.1</u>. <u>Let the general assumptions</u> (1.2) - (1.10) <u>hold</u>. <u>Let</u> $\varepsilon > 0$, $\lambda > 0$ <u>be fixed</u>. <u>Then for every</u> $F \in L^2_{loc}(0,\infty; H)$ <u>and</u> $u_0 \in D(\varphi)$, (2.11) <u>has a unique solution</u> u_λ <u>on</u> $[0,\infty)$ <u>in the sense that</u>

$$u_\lambda \in C([0,\infty); H), \; u_\lambda' \in L^2_{loc}(0,\infty; H)$$

$$u_\lambda(t) \in D(B) \text{ a.e. on } (0,\infty)$$

$$u_\lambda \text{ satisfies (2.11) a.e. on } [0,\infty) .$$

<u>Sketch of proof of Corollary 2.1</u>. Define $G(u)$ by setting

$$G(u)(t) = F(t) - \varepsilon A_\lambda u(t) - a * A_\lambda u(t)$$

and note that (2.11) may be written as

$$u_\lambda' + Bu_\lambda \ni G(u_\lambda), \; u_\lambda(0) = u_0 .$$

Since A_λ is Lipschitz with constant $1/\lambda$, one easily verifies that G has the properties (2.3) and (2.4) for any $T_0 > 0$. Thus the result follows from the second part of Lemma 2.1. (Observe we have not used all of the general assumptions; (1.3), (1.6), $a \in L^1_{loc}[0,\infty)$ are sufficient.)

The next task is to derive bounds on solutions of equations of the form (2.11) (or (1.1)). This we do in some generality. See the end of this section for further remarks.

Proposition 2.1. Let $T > 0$, $D = \partial\Phi$, $C = \partial\Psi$ where $\Phi, \Psi : H \to (-\infty, \infty]$ are convex, l.s.c. and proper. Let $\alpha, \beta, c_0 \in [0, \infty)$, $F \in W^{1,1}(0, T; H)$, $u_0 \in D(\Phi) \cap D(\Psi)$, $a : [0,\infty) \to \mathbb{R}$ be given such that

$$
(2.12) \quad \begin{cases} \text{(i)} \ \Phi(u) \geq -c_0(|u| + 1), \ \Psi(u) \geq -c_0(|u| + 1) \ \text{ for } \ u \in H. \\ \text{(ii)} \ (v, w) \geq \alpha|v|^2 - \beta(|w|^2 + |u|^2 + 1) \ \text{ for all } \ u \in H \ \text{ and } \\ \qquad v \in Cu, \ w \in Du. \\ \text{(iii)} \ a \ \text{ satisfies Conditions (a).} \end{cases}
$$

Then there is a constant C depending only on $|u_0|$, $T, a, \Phi(u_0)$, $\Psi(u_0)$, β, c_0 and $\|F\|_{W^{1,1}(0, T; H)}$ (but not otherwise on Φ, Ψ and not on α) such that if

$$
(2.13) \quad \begin{cases} \text{(i)} \ u, u', v, w \in L^2(0, T; H), \ u(0) = u_0 \\ \text{(ii)} \ v(t) \in Cu(t), \ w(t) \in Du(t) \ \text{ a.e. } \ 0 \leq t \leq T, \\ \text{(iii)} \ u'(t) + w(t) + a * v(t) = F(t) \ \text{ a.e. } \ 0 \leq t \leq T, \end{cases}
$$

then

$$
\max\{\int_0^T |u'(s)|^2 ds, \int_0^T |w(s)|^2 ds, \alpha \int_0^T |v(s)|^2 ds, |u(t)|,
$$

$$
|\Phi(u(t))|, |\Psi(u(t))|, |\int_0^t v(s)ds|\} \leq C
$$

for $0 \leq t \leq T$.

The proof of Proposition 2.1 is given next. The reader may want to skip ahead to the proofs of Theorems 1 and 2 which follow.

Proof of Proposition 2.1. Although the statement of the result is somewhat complicated, the basic idea of the proof is simple. One inner-products (2.13) (iii) with each of $v(t)$ and $u'(t)$, integrates the results over $[0,t]$ and manipulates. (The reader will probably find it helpful to first trace the proof below assuming $\beta = 0$ and make the considerable simplifications which result.)

We will use $c_1, c_2, \ldots$ etc., to denote various constants depending only on a, T, $|u_0|$, $\Phi(u_0)$, $\Psi(u_0)$, β, c_0 and $\|F\|_{W^{1,1}(0,T;H)}$. All estimates below are for $0 \le t \le T$. We will also use estimates of the following sort frequently and without comment:

$\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$, $(x+y)^2 \le 2(x^2+y^2)$, $xy \le \frac{1}{2\eta}x^2 + \frac{\eta}{2}y^2$, and

$\int_0^t |f(s)|ds \le \sqrt{t}\left(\int_0^t |f(s)|^2 ds\right)^{1/2}$ for $x, y, \eta \in (0,\infty)$ and $f \in L^2(0,T;H)$.

Forming the inner-product of (2.13) (iii) with v, integrating over $[0,t]$ and using (2.12) (ii) and [5, Lemma 3.3] yields

$$(2.14)\quad \begin{cases} \Psi(u(t)) - \Psi(u_0) + \alpha \int_0^t |v(s)|^2 ds \\ \qquad + \int_0^t (a * v(s), v(s))ds \le \int_0^t (F(s), v(s))ds \\ \qquad\qquad + \beta\left[\int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds + 1\right]. \end{cases}$$

Next observe that

$$(2.15)\quad \begin{cases} \int_0^t (F(s), v(s))ds = (F(t), \int_0^t (v(\tau)d\tau) - \int_0^t (F'(s), \int_0^s v(\tau)d\tau)ds \\ a * v(t) = \int_0^t a(t-s)v(s)ds = a(0)\int_0^t v(\tau)d\tau + \int_0^t a'(t-s)\int_0^s v(\tau)d\tau ds \ . \end{cases}$$

Hence if

$$(2.16)\qquad g_v(t) = \max_{0 \le s \le t} |\int_0^s v(\tau)d\tau|$$

we have

$$(2.17)\quad \begin{cases} \text{(i)} \quad |\int_0^t (F(s), v(s))ds| \le c_1 g_v(t) \\ \text{(ii)} \quad |a * v(t)| \le c_1 g_v(t) \ . \end{cases}$$

Invoking (2.12) (i) and employing (2.17) (i) in (2.14) yields

$$(2.18)\quad \begin{cases} \int_0^t (a * v(s), v(s))ds + \alpha \int_0^t |v(s)|^2 ds \\ \le c_2(|u(t)| + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds + 1 + g_v(t)) \ . \end{cases}$$

Since $\alpha \ge 0$, Condition (a), (2.16), (2.18) and the monotonicity of

$t \to \|u\|_{L^\infty(0,t;H)} + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds$ imply

$$(2.19)\quad |\int_0^t v(s)ds| \le c_3(1 + \sqrt{\|u\|_{L^\infty(0,t;H)} + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds}) \ .$$

Now by (2.13) (iii) and (2.17) (ii)

$$(2.20)\qquad |w(s)| = |F(s) - (u'(s) + a * v(s))| \leq c_4(1 + |u'(s)| + g_v(s)) .$$

Thus, from (2.19), (2.20) and (2.16)

$$(2.21)\quad g_v(t) \leq c_5\Big(1 + \sqrt{\|u\|_{L^\infty(0,t;H)} + \int_0^t (|u(s)|^2 + g_v(s)^2 + |u'(s)|^2 ds)}\Big) .$$

Next multiply (2.13) (iii) by u' and integrate over $(0,t]$ to find

$$(2.22)\qquad \int_0^t |u'(s)|^2 ds + \Phi(u(t)) - \Phi(u_0) + \int_0^t (a * v(s), u'(s))ds$$

$$= \int_0^t (F(s), u'(s))ds \leq \big(\max_{0 \leq s \leq t} |F(s)|\big) \int_0^t |u'(s)| ds .$$

Calling on (2.17) (ii) and (2.12) (i) again, (2.22) implies

$$(2.23)\qquad \int_0^t |u'(s)|^2 ds \leq c_6\Big(1 + (1 + g_v(t)) \int_0^t |u'(s)| ds + |u(t)|\Big) .$$

The next step is to eliminate the terms involving u in (2.21) and (2.23).

One has

$$(2.24)\qquad |u(t)| = \Big|u_0 + \int_0^t u'(s)ds\Big| \leq |u_0| + \sqrt{t}\,\Big(\int_0^t |u'(s)|^2 ds\Big)^{1/2}$$

$$\leq |u_0| + \frac{t}{2\eta} + \frac{\eta}{2} \int_0^t |u'(s)|^2 ds$$

for $\eta > 0$. Hence

$$(2.25)\qquad \|u\|_{L^\infty(0,t;H)},\ \int_0^t |u(s)|^2 ds \leq c_8\Big(1 + \int_0^t |u'(s)|^2 ds\Big) .$$

Thus from (2.21) and (2.25)

$$g_v(t)^2 \le c_9(1 + \int_0^t g_v(s)^2 ds + \int_0^t |u'(s)|^2 ds) , \tag{2.26}$$

while choosing η so $\eta c_6 < \frac{1}{2}$, (2.23) and (2.24) yield (using also $\int_0^t |u'(s)|ds \le \frac{t}{2\eta} + \frac{\eta}{2} \int_0^t |u'(s)|^2 ds$)

$$\int_0^t |u'(s)|^2 ds \le c_{10}(1 + g_v(t) \int_0^t |u'(s)|ds) . \tag{2.27}$$

The Gronwall inequality, (2.26), $g_v(0) = 0$ and the fact that $t \to \int_0^t |u'(s)|^2 ds$ is nondecreasing imply that

$$g_v(t)^2 \le c_{11}(1 + \int_0^t |u'(s)|^2 ds) . \tag{2.28}$$

Finally, (2.27) and (2.28) give us

$$\int_0^t |u'(s)|^2 ds \le c_{12}(1 + (\int_0^t |u'(s)|^2 ds)^{1/2} \int_0^t |u'(s)|ds)$$

$$\le c_{12}(1 + \frac{\eta}{2} \int_0^t |u'(s)|^2 ds + \frac{1}{2\eta}(\int_0^t |u'(s)|ds)^2) .$$

Appropriate choice of η implies

$$\int_0^t |u'(s)|^2 ds \le c_{13}(1 + (\int_0^t |u'(s)|ds)^2) . \tag{2.29}$$

To see that (2.29) implies a bound on $\int_0^t |u'(s)|^2 ds$, proceed as follows: Assume $t_0 \ge 0$ and $\int_0^{t_0} |u'(s)|^2 ds \le M$. Then $\int_0^{t_0} |u'(s)|ds \le \sqrt{t_0}\sqrt{M}$

and, from (2.29)

$$\int_{t_0}^{t} |u'(s)|^2 ds \le \int_0^t |u'(s)|^2 ds \le c_{13}(1 + (\sqrt{t_0}\sqrt{M} + \int_{t_0}^{t} |u'(s)| ds)^2)$$

$$\le c_{13}(1 + 2t_0 M + 2(t - t_0) \int_{t_0}^{t} |u'(s)|^2 ds)$$

so

$$\int_{t_0}^{t} |u'(s)|^2 ds \le \frac{1}{1 - 2c_{13}(t - t_0)} c_{13}(1 + 2t_0 M) \le 2c_{13}(1 + 2t_0 M)$$

for $2c_{13}(t - t_0) \le \frac{1}{2}$. Iterating, we bound $\int_0^t |u'(s)|^2 ds$. Since $t_0 = 0$, $M = 0$ may be used to start, (2.29) implies

$$\int_0^t |u'(s)|^2 ds \le c_{14} . \tag{2.30}$$

The proof of Proposition 2.1 is essentially complete. First (2.30), (2.24) and (2.28) imply $g_v(t)$, $|u(t)| \le c_{15}$. This information, (2.17) (ii), (2.12) (i) and (2.22) imply $|\Phi(u(t))| \le c_{16}$. Since $w = F - (u' + a * v)$, $\int_0^t |w(s)|^2 ds \le c_{17}$. All these estimates, (2.18) and the bound on $|\int_0^t (a * v(s), v(s)) ds|$ supplied by Condition (a) imply $\alpha \int_0^t |v(s)|^2 ds \le c_{18}$. Finally, $|\Psi(u(t))|$ is bounded via (2.12) (i) and (2.14). The proof is complete.

<u>Proof of Theorems 1 and 2</u>. The first step in both Theorems is to let $\lambda \downarrow 0$ in (2.11) with $\varepsilon \in (0,1]$ fixed. Now $B + \varepsilon A_\lambda = \partial(\varphi + \varepsilon\psi_\lambda)$ and $A_\lambda = \partial\psi_\lambda$ where

$\psi_\lambda(x) = \min\{\psi_H(y) + (2\lambda)^{-1}|y - x|^2 : y \in H\}$ is as in [5, Prop. 2.11]. Since convex functions are bounded below by affine functions, there exists c_0 such that $\varphi + \varepsilon\psi_\lambda$ and ψ_λ are bounded below by $-c_0(|u| + 1)$ uniformly for $\varepsilon, \lambda \in [0,1]$. Set $\Phi = \varphi + \varepsilon\psi_\lambda$, $\Psi = \psi_\lambda$ in Proposition 2.1. In view of (1.7) we have (2.12) (ii) where ε may be used as the coefficient α of $|v|^2$ in (2.12) (ii). Hence by Proposition 2.1 for $T > 0$ there is a C_T independent of $\varepsilon, \lambda \in (0,1]$ such that for $\lambda, \varepsilon \in (0,1]$ and $t \in [0,T]$

$$
(2.31)\quad \begin{cases}
\text{(i)} & |\psi_\lambda(u_\lambda(t))| \leq C_T \\
\text{(ii)} & |\varphi(u_\lambda(t))| \leq C_T \\
\text{(iii)} & |u_\lambda(t)| \leq C_T \\
\text{(iv)} & \varepsilon \int_0^t |A_\lambda u_\lambda(s)|^2 ds \leq C_T \\
\text{(v)} & \int_0^t |u_\lambda'(s)|^2 ds \leq C_T \\
\text{(vi)} & \int_0^t |F(s) - (u_\lambda'(s) + a * A_\lambda u_\lambda(s)|^2 ds \leq C_T .
\end{cases}
$$

The compactness condition (1.8), and (2.31 (ii), (iii) imply that there is a compact subset K_T of W for which $u_\lambda([0,T]) \subseteq K_T$. Hence $u_\lambda \in C([0,T]; H)$ implies $u_\lambda \in C([0,T]; W)$ (see Lemma 2.2 below). Since $|u_\lambda(t) - u_\lambda(s)| \leq \sqrt{t - s}(\int_s^t |u_\lambda'(\tau)|^2 d\tau)^{1/2}$, the functions u_λ are equicontinuous on bounded subsets of $[0,\infty)$ with values in H. From

$u_\lambda([0,T]) \subseteq K_T$, they are also equicontinuous with values in W (see Lemma 2.2 below). Then by the weak sequential compactness of closed balls in $L^2(0,T;H)$ for $T>0$, the Ascoli theorem and (2.31) (iv), (v) and (vi) we have the existence of functions $u_\varepsilon \in C([0,\infty);W)$, $v_\varepsilon, w_\varepsilon \in L^2_{loc}([0,\infty);H)$ with $u'_\varepsilon \in L^2_{loc}([0,\infty);H)$ and a sequence $\lambda_n \downarrow 0$ such that

$$(2.32)\quad \begin{cases} \text{(i)}\ u_{\lambda_n} \to u_\varepsilon \ \text{ in } C([0,\infty);W) \\ \left.\begin{array}{l} \text{(ii)}\ A_{\lambda_n} u_{\lambda_n} \to v_\varepsilon \\ \text{(iii)}\ u'_{\lambda_n} \to u'_\varepsilon \\ \text{(iv)}\ F - (u'_{\lambda_n} + \varepsilon A_{\lambda_n} u_{\lambda_n} + a * A_{\lambda_n} u_{\lambda_n}) \to w_\varepsilon \end{array}\right\} \text{weakly in } L^2(0,T;H) \text{ for } T>0 . \end{cases}$$

In particular, (2.32) (i) implies $u_{\lambda_n} \to u_\varepsilon$ in $L^2(0,T;H)$ for $T>0$. By the demiclosed property of maximal monotone operators (and [5, Example 2.3.3]) $v_\varepsilon(t) \in A_H u_\varepsilon(t)$ a.e. and $w_\varepsilon(t) \in Bu_\varepsilon(t)$ a.e. (since $F - (u'_{\lambda_n} + \varepsilon A_{\lambda_n} u_{\lambda_n} + a * A_{\lambda_n} u_{\lambda_n}) \in Bu_{\lambda_n}$ a.e.). Since $v \to a * v$ is bounded, and linear on $L^2(0,T;H)$ it is weakly continuous and $a * A_{\lambda_n} u_{\lambda_n} \to a * v_\varepsilon$ weakly in $L^2(0,T;H)$. Thus (2.32) implies

$$(2.33)\quad \begin{cases} \text{(i)}\ u'_\varepsilon + w_\varepsilon + \varepsilon v_\varepsilon + a * v_\varepsilon = F \\ \text{(ii)}\ u'_\varepsilon, w_\varepsilon, v_\varepsilon \in L^2_{loc}(0,\infty;H), w_\varepsilon(t) \in Bu_\varepsilon(t), v_\varepsilon(t) \in A_H u_\varepsilon(t) \ \text{ a.e. } (0 \le t < \infty). \end{cases}$$

Now we want to let $\varepsilon \downarrow 0$. Invoking Proposition 2.1 again we conclude that $\psi(u_\varepsilon(t))$, $\varphi(u_\varepsilon(t))$, $|u_\varepsilon(t)|$, $\varepsilon \int_0^t |v_\varepsilon(s)|^2 ds$, $\int_0^t |u'_\varepsilon(s)|^2 ds$, $\int_0^t |w_\varepsilon(s)|^2 ds$ and $|\int_0^t v_\varepsilon(s) ds|$ are all bounded uniformly for $\varepsilon \in (0,1]$, $t \in [0,T]$.

If (1.12) holds we obtain from these estimates that also $\int_0^t |v_\varepsilon(s)|^2 ds$ is locally bounded uniformly for $\varepsilon \in (0,1]$, and the passage to the limit as $\varepsilon \downarrow 0$ may be done exactly as above. This proves Theorem 2.

In the case of Theorem 1 the situation is different for we no longer have an estimate on $\int_0^t |v_\varepsilon(s)|^2 ds$ independent of $\varepsilon \in (0,1]$, which is where the assumption in Theorem 1 that $A : W \to W'$ is everywhere defined and single-valued comes in to play. We write $v_\varepsilon = Au_\varepsilon$ (we may use A rather than A_H by (1.6)) in this case). Just as above, we have the existence of a sequence $\varepsilon_n \downarrow 0$, $u \in C([0,\infty); W)$, $u' \in L^2_{loc}([0,\infty); H)$, $w \in L^2_{loc}([0,\infty); H)$ such that

$$
(2.34)\quad \begin{cases} \text{(i)} & u_{\varepsilon_n} \to u \ \text{ in } \ C([0,\infty); W),\ u'_{\varepsilon_n} \to u' \ \text{ weakly in } \ L^2_{loc}([0,\infty); H)\,, \\ \text{(ii)} & \varepsilon_n Au_{\varepsilon_n} \to 0 \ \text{ in } \ L^2_{loc}([0,\infty); H)\,, \\ \text{(iii)} & w_{\varepsilon_n} = F - (u'_{\varepsilon_n} + \varepsilon_n Au_{\varepsilon_n} + a * Au_{\varepsilon_n}) \to w \ \text{ weakly in} \\ & \qquad L^2(0,T;H) \quad \text{for} \quad T > 0\,, \end{cases}
$$

($\varepsilon Au_\varepsilon \to 0$ in $L^2_{loc}([0,\infty); H)$ since $\varepsilon \int_0^T |Au_\varepsilon(s)|^2 ds < c_T$ for $0 \le T$, $0 < \varepsilon \le 1$). To take the limit of Au_{ε_n} we use that since A is maximal monotone, single-valued and everywhere defined it is necessarily continuous from the strong to the weak topology and is bounded in some neighborhood of the compact set $u([0,T])$ in W (see, e.g., [17]). Thus $Au_{\varepsilon_n}(t) \to Au_\lambda(t)$ weakly in W' and boundedly for bounded t. Hence $Au_{\varepsilon_n} \to Au_\lambda$ weakly in $L^2(0,T;W')$ for $T > 0$ and $a * Au_{\varepsilon_n} \to a * Au_\lambda$

weakly in $L^2(0,T; W')$. We conclude that $u' + w + a * Au = F$ a.e. where $w(t) \in Bu(t)$ a.e., as desired. Finally, we use the bound on $|\int_0^t Au_\varepsilon(s)ds|$ provided by Proposition 2.1. Clearly $\int_0^t Au_{\varepsilon_n}(s)ds \to \int_0^t Au(s)ds$ weakly in W' for $t \geq 0$. Since W is dense in H and $\int_0^t Au_{\varepsilon_n}(s)ds$ is bounded in H, $\int_0^t Au(s)ds \in H$ and $\int_0^t Au_{\varepsilon_n}(s)ds \to \int_0^t Au(s)ds$ weakly in H as well as in W'. The proof is complete.

It remains to prove:

<u>Lemma 2.2</u>. <u>Let</u> X, Y, K <u>be metric spaces where</u> K <u>is compact</u>. <u>Let</u>

(i) $\mathfrak{F}$ <u>be a set of maps</u> $f: X \to K$,

(ii) g <u>be a one-to-one continuous mapping of</u> K <u>into</u> Y,

(iii) $\{g \circ f : f \in \mathfrak{F}\} : X \to Y$ <u>be an equicontinuous family</u>.

<u>Then</u> $\mathfrak{F}$ <u>is equicontinuous</u>.

<u>Proof</u>. Let $h = g^{-1}$. Then $h: g(K) \to K$ is continuous (since K is compact and g is continuous) and therefore uniformly continuous (since $g(K)$ is compact). Now $g \circ \mathfrak{F} = \{g \circ f : f \in \mathfrak{F}\}$ is equicontinuous by assumption and $\mathfrak{F} = h \circ (g \circ \mathfrak{F})$ is therefore also equicontinuous. (This lemma, formulated for us by W. Rudin, is used with K a compact set in W, g the injection $W \to H$, $X = [0,T]$ or $[0,\infty)$ and $Y = H$ in the current work.)

<u>Remarks on (1.7) and Proposition (2.1)</u>. We wish to mention here that our conditions and arguments allow various kinds of perturbations. For example, consider the perturbed problem

$$(1.1)_P \qquad \begin{cases} u' + Bu + a * A_P u \ni F(t) \\ u(0) = u_0 \end{cases}$$

where $A_P = A + P$, A and B satisfy the general conditions and the perturbation $P : H \to H$ is Lipschitz continuous, i.e. there is an $\omega \in [0,\infty)$ such

$$(2.35) \qquad |Px_1 - Px_2| \le \omega|x_1 - x_2| \quad \text{for } x_1, x_2 \in H .$$

Then $A_H + P + \omega I$ is monotone and $A_{P\lambda}$, $J_{P\lambda}$ are well-defined by

$$x + \lambda(A_H x + Px) \ni u \Longrightarrow x = J_{P\lambda}u, \; A_{P\lambda} = \lambda^{-1}(I - J_{P\lambda})$$

for $0 < \lambda < 1/\omega$. Moreover, it is an exercise to show that

$|A_{P\lambda}u - A_\lambda u| \le \omega((1-\lambda\omega)^{-1}|u - (x_0 + \lambda(y_0 + Px_0)| + |Bx_0|)$ for $0 < \lambda < 1/\omega$, $x_0 \in D(A_H)$, $y_0 \in Ax_0$. Thus if A_λ satisfies (1.7), so will $A_{P\lambda}$ for small $\lambda > 0$ (with another choice of β). Thus (1.7) is stable under Lipschitz continuous perturbations in particular. Hence we can hope to treat $(1.1)_P$ as we did (1.1).

If P is not itself a gradient, it is probably more convenient to approximate $(1.1)_P$ via

$$u'_\lambda + Bu_\lambda + \varepsilon(A_\lambda u_\lambda + Pu_\lambda) + a * (A_\lambda u_\lambda + Pu_\lambda) \ni F$$

than to use $A_{B\lambda}$ and then proceed as in Proposition 2.1. Additional terms arise from (u'_λ, Pu_λ) when multiplying by $A_\lambda u_\lambda + Pu_\lambda$, but these contribute no new difficulties and the same estimates are obtained. (Clearly $(Bu_\lambda, A_\lambda u_\lambda + Pu_\lambda)$ has the desired form of lower bound when (2.35) and (1.7) hold.)

We will not say more about the many other possibilities here, as it is not very clear at this time in which direction to push the theory.

Section 3. Proof of Theorem 3.

(i) Let $u, u', v, w \in L^2_{loc}([0,\infty); H)$ satisfy

$$(3.1)\qquad \begin{cases} \text{(i)} \quad u' + w + a * v = F \\ \text{(ii)} \quad w(t) \in Bu(t), \ v(t) \in Au(t) \quad \text{a.e.} \quad (0 \le t < \infty) . \end{cases}$$

Form the inner-product of (3.1) (i) with v, integrate over $[0,t]$ and use $(w(t), v(t)) \ge 0$ to find

$$(3.2)\qquad \psi(u(t)) - \psi(u_0) + Q_a(v;t) \le \int_0^t (F(\tau), v(\tau))d\tau ,$$

where

$$(3.3)\qquad Q_a(v,t) = \int_0^t (a * v(s), v(s))ds = \frac{a(t)}{2} \left|\int_0^t v(s)ds\right|^2$$

$$- \frac{1}{2}\int_0^t a'(\tau)\left|\int_0^\tau v(s)ds\right|^2 d\tau - \frac{1}{2}\int_0^t a'(t-\tau)\left|\int_\tau^t v(\sigma)d\sigma\right|^2 d\tau$$

$$+ \frac{1}{2}\int_0^t\int_0^\tau a''(\tau - s)\left|\int_s^\tau v(\sigma)d\sigma\right|^2 ds d\tau .$$

See the proof of Proposition (a) in Appendix (a) concerning the validity of the right-most equality in (3.3). Note each term on the right of (3.3) is nonnegative since a satisfies conditions (a_2). Integrating by parts and using (1.13) we have

$$\int_0^t (F(\tau), v(\tau))d\tau = (F(t), \int_0^t v(s)ds) - \int_0^t (F'(\tau), \int_0^\tau v(s)ds)d\tau$$

$$\le \delta a(t)\left|\int_0^t v(s)ds\right| + \delta\int_0^t |a'(\tau)|\left|\int_0^\tau v(s)ds\right| d\tau$$

$$\le \delta a(t)\left|\int_0^t v(s)ds\right| + \delta\sqrt{a(0)}\left(\int_0^t |a'(\tau)|\left|\int_0^\tau v(s)ds\right|^2 d\tau\right)^{1/2}$$

$$\le \delta^2 a(t) + \frac{a(t)}{4}\left|\int_0^t v(s)ds\right|^2 + \delta^2 a(0) + \frac{1}{4}\int_0^t |a'(\tau)|\left|\int_0^\tau v(s)ds\right|^2 d\tau ,$$

so, from (3.3) and the above

$$(3.4) \qquad \int_0^t (F(\tau), v(\tau))d\tau \le \delta^2(a(0) + a(t)) + \frac{1}{2} Q_a(v;t) .$$

Together (3.2) and (3.4) imply

$$(3.5) \qquad \psi(u(t)) - \psi(u_0) + \frac{1}{2} Q_a(v;t) \le \delta^2(a(0) + a(t)) .$$

Since ψ is bounded below, (3.5) implies

$$(3.6) \qquad \sup_{t\ge 0} Q_a(v,t) < \infty, \quad \sup_{t\ge 0} \psi(u(t)) < \infty .$$

The estimate

$$(3.7) \qquad |a * v(t)| \le 2a(0)\, Q_a(v;t), \quad 0 \le t < \infty$$

follows from Lemma 3.1 which is stated and proved later. Hence (1.14) (a), (b), (c) hold.

(ii) If also $\psi(u) \to \infty$ as $\|u\| \to \infty$, (3.6) implies (1.15) (a). We now seek to bound $\varphi(u(t))$. By $w(t) = F(t) - (u'(t) + a * v(t)) \in \partial\varphi(u(t))$ we have

$$(3.8) \qquad \begin{cases} \text{(i)} \quad \dfrac{d}{dt}\varphi(u(t)) = (F(t) - (u'(t) + a * v(t)),\ u'(t)) \\ \qquad \text{and by the definition of subdifferential} \\ \text{(ii)} \quad \varphi(u(t)) \le \varphi(u_0) + (F(t) - (u'(t) + a * v(t)),\ u(t) - u_0) . \end{cases}$$

From (1.15) (a) we have that $c_2 = \sup_{t\ge 0} |u(t) - u_0| < \infty$ and from (3.6), (3.7) and (1.13), $c_1 = \sup_{t\ge 0} |F(t) - a * v(t)| < \infty$. Hence adding (3.8) (i) and (ii)

$$\frac{d}{dt}\varphi(u(t)) + \varphi(u(t)) \le - |u'(t)|^2 + (c_1 + c_2)|u'(t)| + \varphi(u_0) + c_1 c_2 \le c_3$$

where c_3 is independent of t. Hence (1.15) (b) follows if φ is bounded below.

In order to prove (1.15) (c) we first show that $u \in UC([0,\infty); H)$. Forming the inner-product of (3.1) (i) with $w(t)$ and integrating over $[t, t+1]$ gives the inequality

$$(3.9) \qquad \varphi(u(t+1)) - \varphi(u(t)) + \int_t^{t+1} |w(s)|^2 ds \le c_1 \int_t^{t+1} |w(s)| ds$$

$$\le c_1 [\int_t^{t+1} |w(s)|^2 ds]^{1/2}$$

with the same constant c_1 as above. Because $\varphi(u(t))$ is bounded, (3.9) implies $\int_t^{t+1} |w(s)|^2 ds$ is bounded. Since $u' = F - (a * v + w)$ and $F - a * v \in L^\infty(0,\infty; H)$, we also have $\sup_{0<t<\infty} \int_t^{t+1} |u'(s)|^2 ds < \infty$. Thus $u \in UC([0,\infty); H)$. In conjunction with the previously obtained bounds on $|u(t)|$ and $\varphi(u(t))$, the compactness assumption implies that

$$(3.10) \qquad u([0,\infty)) \text{ is strongly precompact in } W .$$

Hence $u \in UC([0,\infty); W)$ follows from Lemma 2.2, proving (1.15) (c).

(iii) The asymptotic result is obtained by reducing the analysis to the scalar case as follows. We now have $v(t) = Au(t)$ and A is locally bounded. Thus (3.10) implies

$$(3.11) \qquad \sup_{0 \le t < \infty} \|Au(t)\|_{W'} < \infty .$$

The demicontinuity of A, together with $u \in UC([0,\infty); W)$, (3.10), (3.11), implies that for any $z \in W$ the function e defined by

$$(3.12) \qquad e(t) = \langle Au(t), z \rangle ,$$

satisfies

$$e(t) \in UC[0,\infty), \quad \sup_{0 \le t < \infty} |e(t)| < \infty . \tag{3.13}$$

Let $T > 0$ be arbitrary and define e_T by

$$e_T(t) = \chi[0,T]e(t), \quad -\infty < t < \infty , \tag{3.14}$$

where χ is the characteristic function. By the arguments in the proof of Lemma 3.1 below (in particular note (3.27)) one shows that

$$Q_a(e;\, T) = \frac{1}{4\pi} \int_{-\infty}^{\infty} |\hat{e}_T(\sigma)|^2 d\alpha(\sigma) , \tag{3.15}$$

where $\hat{}$ denotes the Fourier transform, $\hat{e}_T(\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma t} e_T(t)dt$ and where α is a positive measure satisfying (3.24). By (3.12) one has

$$|\hat{e}_T(\sigma)| \le |\hat{v}_T(\sigma)|\ |w| , \tag{3.16}$$

where $v_T = \chi[0,T]v$, $v = Au$. Invoking conditions a_2, (3.6), (3.15), (3.16) and formula (3.27) yield *the estimate*

$$\tag{3.17} \left\{ \begin{aligned} 0 &\le \sup_{0 \le T < \infty} Q_a(e;T) \le \sup_{0 \le T < \infty} \frac{|z|^2}{4\pi} \int_{-\infty}^{\infty} |\hat{z}_T(\sigma)|^2 d\alpha(\sigma) \\ &= |z|^2 \sup_{0 \le T < \infty} Q_a(Au;\, T) < \infty . \end{aligned} \right.$$

But by conditions a_2, $a'(t) \not\equiv 0$, and Corollary 2.2 of [16] (see also [15]) $a(t)$ is a strongly positive kernel. *This fact together with* (3.13), (3.17) and Theorem 1 (ii) of [16] shows that

$$\lim_{t \to \infty} e(t) = 0 . \tag{3.18}$$

Finally (3.12), (3.18) establish (1.16), which together with (3.10) and $A^{-1}(0)$ a singleton implies (1.17). This completes the proof of Theorem 3.

Remark. Theorem 1 (ii) of [16] can be applied although $e(t)$ only satisfies (3.13), (3.17) and is not a bounded solution of a scalar Volterra equation. This fact, which is evident from the proof of Theorem 1 (ii) of [16], has been exploited and generalized by O. Staffans [19], [20], [21].

The following lemma appears as Lemma 6.2 in Staffans [20] for the real scalar case; it is included here for the convenience of the reader.

Lemma 3.1. *Let a be positive definite on $[0,\infty)$ and $v \in L^2_{loc}([0,\infty); H)$. Then*

$$|a * v(T)|^2 \leq 2a(0)Q_a(v;T), \quad 0 < T < \infty . \tag{3.19}$$

Remark. If a satisfies conditions a_2, then by the identity (3.3) a is positive definite on $[0,\infty)$. See also [16; Theorem 2] and remarks immediately following.

Proof of Lemma 3.1. Extend a evenly by

$$a(-t) = a(t), \quad 0 \leq t < \infty , \tag{3.20}$$

and let,

$$v_T = \chi[0,T]v . \tag{3.21}$$

Then by (3.20), (3.21), Fubini's theorem and some elementary calculations one obtains

$$Q_a(v;T) = \frac{1}{2}\int_{-\infty}^{\infty} a(s)k(s)ds, \quad 0 < T < \infty , \tag{3.22}$$

where

$$(3.23) \qquad k(s) = \int_{-\infty}^{\infty} (v_T(t - s),\ v_T(t))dt\ .$$

Since a is positive definite on $(-\infty, \infty)$, Bochner's theorem [18] implies the existence of a positive measure α such that

$$(3.24) \qquad a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sigma t} d\alpha(\sigma), \quad -\infty < t < \infty, \quad \int_{-\infty}^{\infty} d\alpha(\sigma) < \infty\ .$$

Combining (3.22) with the first part of (3.24) yields

$$(3.25) \qquad Q_a(v;T) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{k}(-\sigma) d\alpha(\sigma), \quad 0 < T < \infty\ .$$

But (3.23) and Parseval's theorem give

$$(3.26) \qquad \hat{k}(\sigma) = |\hat{v}_T(\sigma)|^2 = \hat{k}(-\sigma), \quad -\infty < \sigma < \infty\ .$$

From (3.25), (3.26) one has

$$(3.27) \qquad Q_a(v;T) = \frac{1}{4\pi} \int_{-\infty}^{\infty} |\hat{v}_T(\sigma)|^2 d\alpha(\sigma), \quad 0 < T < \infty\ .$$

The conclusion (3.19) is now a result of the following elementary calculation which uses (3.20), (3.21), (3.24), (3.27), Fubini's theorem, and Schwartz's inequality:

$$|a * v(T)|^2 = |\int_{-\infty}^{\infty} a(T-\tau)v_T(\tau)d\tau|^2$$

$$\leq \sup_{-\infty<t<\infty} |\int_{-\infty}^{\infty} a(t-\tau)v_T(\tau)d\tau|^2$$

$$= \sup_{-\infty<t<\infty} \frac{1}{4\pi^2} |\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} e^{\sigma i(t-\tau)} d\alpha(\sigma))v_T(\tau)d\tau|^2$$

$$= \sup_{-\infty<t<\infty} \frac{1}{4\pi^2} |\int_{-\infty}^{\infty} e^{i\sigma t}\hat{v}_T(\sigma)d\alpha(\sigma)|^2$$

$$\leq \frac{1}{4\pi^2}(\int_{-\infty}^{\infty} d\alpha(\sigma)) \int_{-\infty}^{\infty} |\hat{v}_T(\sigma)|^2 d\alpha(\sigma) = 2a(0)Q_a(v;T) .$$

This completes the proof of Lemma 3.1.

Remarks. The above analysis may be extended to cover the solutions of (1.1) obtained in the proof of Theorem 1 as follows: First let (1.7) hold with $\beta = 0$. The estimate (3.5) will hold, with the same proof, with u replaced by u_ε and v by $v_\varepsilon = A_H u_\varepsilon$, where $u_\varepsilon, v_\varepsilon$ are as in the proof of Theorem 1. Hence the analogue of (3.6) holds uniformly in $\varepsilon > 0$. Using the identity (3.3) with $v = v_\varepsilon = Au_\varepsilon$, letting $\varepsilon_n \to 0$, using the convergence $\int_0^t Au_{\varepsilon_n}(s)ds \to \int_0^t Au(s)ds$ weakly in H for every t and invoking Fatou's lemma we find that

$$Q_a(Au;t) = \frac{a(t)}{2} |\int_0^t Au(s)ds|^2 - \frac{1}{2}\int_0^t a'(\tau)|\int_0^\tau Au(s)ds|^2 d\tau$$

$$- \frac{1}{2}\int_0^t a'(t-\tau) |\int_\tau^t Au(\sigma)d\sigma|^2 d\tau$$

$$+ \frac{1}{2}\int_0^t \int_0^\tau a''(\tau - s)|\int_s^\tau Au(\sigma)d\sigma|^2 ds d\tau$$

is bounded independent of $t \geq 0$. (Here we regard the above expression as the definition of $Q_a(Au;t)$.) We also have $\psi(u(t)) \leq \lim_{n\to\infty} \psi_{\varepsilon_n}(u_{\varepsilon_n}(t))$ by properties of ψ_ε. In this way, we can preserve all of the assertions of Theorem 3 except (1.14) (b) which is replaced by

$$|a * Au(t)|^2 \leq \liminf_{n\to\infty} 2a(0)Q_a[Au_{\varepsilon_n};t] .$$

(Similarly, $Q_a(v;T)$ is replaced by $\liminf_{n\to\infty} Q_a(Au_{\varepsilon_n};t)$ in (3.17).)

Section 4. Proof of Theorem 4.

Let us first observe that for $v \in L^2_{loc}([0,\infty) : H)$ and $a \in L^1_{loc}([0,\infty))$, $T^* \geq T$, $v_T = \chi(0,T)v$, a_{T^*} the even extension of $\chi([0,T^*])a$ to $(-\infty,\infty)$, we have

$$(4.1) \qquad Q_a(v;T) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} a_{T^*}(t-s)v_T(s)ds, v_T(t)\right)dt$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}(\hat{a}_{T^*}(\sigma)\hat{v}_T(\sigma), \hat{v}_T(\sigma))d\sigma$$

where the first equality follows from elementary manipulations, and the second from Parseval's equality and $\hat{}$ is as in Section 3. Hence

$$(4.4) \qquad Q_a(v;T) \geq \frac{1}{2}\left(\inf_{\infty > \sigma > -\infty} \hat{a}_{T^*}(\sigma)\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{v}_T(\sigma)|^2 d\sigma$$

$$= \frac{1}{2}\left(\inf_{\infty > \sigma > -\infty} \hat{a}_{T^*}(\sigma)\right)\int_0^T |v(s)|^2 ds .$$

Since $a_{T^*}(\sigma) = 2\int_0^{T^*} \cos(\sigma t)a(t)dt$, (4.4) and (1.18) imply that

$$(4.5) \qquad Q_a(v;T) \geq (\delta - \alpha)\int_0^T |v(s)|^2 ds .$$

Now inner-product $u' + w + a * v = F$ with v, integrate from 0 to T and use (1.18) (i), (4.5) to find

$$(4.6) \qquad \psi(u(T)) - \psi(u(0)) + \delta\int_0^T |v(s)|^2 ds$$

$$\leq \psi(u(T)) - \psi(u(0)) + \int_0^T (w(s), v(s))ds + Q_a(v;T)$$

$$= \int_0^T (F(s), v(s))ds \leq K\left(\int_0^T |v(s)|^2 ds\right)^{1/2}$$

where $K^2 = \int_0^\infty |F(s)|^2 ds$. From (4.6) and $\inf \psi > -\infty$, we deduce that $\int_0^T |v(s)|^2 ds$ and $\psi(u(T))$ are bounded independent of $T \geq 0$, and the results follow.

5. Infinite Delay

To treat $(1.1)_\infty$ by means of Theorems 1-4 we observe that it may be formally rewritten as

(5.1) $$u' + Bu + a * Au \ni F(t)$$

where

(5.2) $$F(t) = f(t) - p(t)$$

and $p(t) = \int_{-\infty}^{0} a(t - s)Au(s)ds = \int_{-\infty}^{0} a(t - s)Ah(s)ds$. Assuming that $z(t) \in Ah(t)$ a.e., $-\infty < t \leq 0$, we are reduced to considering properties of

(5.3) $$p(t) = \int_{-\infty}^{0} a(t - s)z(s)ds .$$

For the existence theorems we want $F \in W^{1,1}_{loc}([0,\infty); H)$. This will be the case if $f, p \in W^{1,1}_{loc}([0,\infty); H)$.

Proposition 5.1. *Let* $z \in L^1(-\infty, 0; H)$, a *be locally absolutely continuous and*

(5.4) $$\sup_{0 \leq t < \infty} \int_{t}^{t+T} (|a(s)| + |a'(s)|)ds = K_T < \infty \quad (T > 0) .$$

Then p *given by* (5.3) *is in* $W^{1,1}_{loc}([0,\infty); H)$.

Proof. First, by Fubini's theorem and (5.4)

$$\int_0^T \int_{-\infty}^{0} |a(t - s)z(s)|dsdt = \int_{-\infty}^{0} \int_{-s}^{T-s} |a(\tau)|d\tau\, |z(s)|ds \leq K_T \int_{-\infty}^{0} |z(s)|ds ,$$

which shows that the integral defining p in (5.3) converges for a.e. $t \geq 0$ and $p \in L^1_{loc}([0,\infty); H)$. Also, by Fubini's theorem,

$p'(t) = \int_{-\infty}^{0} a'(t-s)z(s)ds$ and lies in $L^1_{loc}([0,\infty); H)$ by the same calculation as above.

For application of Theorem 3 we use:

<u>Proposition 5.2</u>. <u>Let</u> $z \in L^1(-\infty, 0; H)$ <u>and</u> a <u>satisfy conditions</u> $(a)_2$. <u>Then the function</u> p <u>defined by</u> (5.3) <u>satisfies</u>

$$|g(t)| \leq \delta a(t), \quad |g'(t)| \leq -\delta a'(t) \qquad (0 \leq t < \infty) \tag{5.5}$$

<u>where</u> $\delta = \int_{-\infty}^{0} |z(s)|ds$.

<u>Proof</u>. By conditions $(a)_2$ and $z \in L^1(-\infty, 0; H)$ one has

$$|p(t)| \leq \int_{-\infty}^{0} a(t-s)|g(s)|ds \leq a(t) \int_{-\infty}^{0} |g(s)|ds \qquad (0 \leq t < \infty)$$

and

$$|p'(t)| \leq -a'(t) \int_{-\infty}^{0} |z(s)|ds \qquad (0 \leq t < \infty) .$$

Thus (5.5) holds and the proof is complete.

For applications of Theorem 4, take $f \in L^2(0, \infty; H)$ and recall Young's inequality:

$$\left\| \int_{-\infty}^{0} a(t-s)z(s)ds \right\|_{L^2(0,\infty;H)} \leq \|a\|_{L^p(0,\infty)} \|z\|_{L^q(-\infty;0;H)}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{3}{2}$, $1 \leq p, q < \infty$. For example, if $a \in L^1(0,\infty)$, $f \in L^2(0,\infty; H)$, then it suffices to have $z \in L^2(0,\infty; H)$.

6. Examples

Example 1

We begin with a brief outline of problems of the form of [2, Example 1]. Let Ω be a bounded open domain in $\mathbb{R}^N$ with smooth boundary Γ and consider the integro-differential equation

$$(6.1) \qquad u_t(t,x) - \Delta u(t,x) + \int_0^t a(t-s)g(u(s,x))ds = F(t,x)$$

for $(t,x) \in (0,\infty) \times \Omega$, together with the boundary condition

$$(6.2) \qquad -\frac{\partial u}{\partial n} \in \gamma(u) \quad \text{a.e. on} \quad (0,\infty) \times \Gamma$$

and the initial condition

$$(6.3) \qquad u(0,x) = u_0(x), \; x \in \Omega .$$

In what follows we formulate conditions which imply that various of our assumptions hold. If γ is maximal monotone in $\mathbb{R}$, $0 \in \gamma(0)$, $\gamma = \partial j$ where $j : \mathbb{R} \to [0,\infty]$ is convex, l.s.c. and

$$(6.4) \qquad \begin{cases} \varphi(u) = \frac{1}{2}\int_\Omega |\mathrm{grad}\, u|^2 dx + \int_\Gamma j(u)dx \\ D(\varphi) = \{u \in H^1(\Omega),\; j(u) \in L^1(\Gamma)\}, \end{cases}$$

then $\varphi : H = L^2(\Omega) \to (-\infty,\infty]$ is convex, l.s.c. and

$$(6.5) \quad \partial\varphi(u) = -\Delta u \quad \text{for } u \in D(\partial\varphi) = \{u \in H^2(\Omega),\; -\frac{\partial u}{\partial n} \in \gamma(u), \quad \text{a.e. on } \Gamma\} .$$

See [6]. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy

$$(6.6) \qquad g \in C(-\infty,\infty),\; g \text{ nondecreasing}, \quad g(0) = 0$$

and

$$(6.7) \qquad |g(u)| \le c_1(|u|^{p-1} + 1) \quad \text{for} \quad u \in \mathbb{R}$$

for some constant c_1 and where p satisfies

$$(6.8) \qquad \begin{cases} 2 \le p < \infty & \text{if } N = 1, 2 \\ 2 \le p < \dfrac{2N}{N-2} & \text{if } N \ge 3 . \end{cases}$$

Setting $W = L^p(\Omega)$, $G(u) = \int_0^u g(r)dr$ and $\psi(u) = \int_\Omega G(u(x))dx$ for $u \in W$ we have that $\psi : W \to (-\infty, \infty)$ is continuous and convex (by (6.6), (6.7)) while $u \to \partial\psi(u) = g(u)$ is continuous from $W = L^p(\Omega)$ into $W' = L^{p/(p-1)}(\Omega)$ in the strong topologies. Moreover by (6.4) and $j(u) \ge 0$, $\{u \in H; |\varphi(u)| + |u| \le K\}$ is bounded in $H^1(\Omega)$ and therefore by (6.8) and the imbedding theorems is compact in $W = L^p(\Omega)$. Also, using Fatou's lemma one can show that ψ_H in (1.4) is given by

$$\psi_H(u) = \begin{cases} \int_\Omega G(u(x))dx, & u \in L^2(\Omega),\ G(u) \in L^1(\Omega) \\ +\infty & , u \in L^2(\Omega),\ G(u) \notin L^1(\Omega) . \end{cases}$$

In particular, (1.5) holds. One also has

$$A_H u = g(u), \quad D(A_H) = \{u \in L^2(\Omega); g(u) \in L^2(\Omega)\}$$

and it is straightforward to show, using $0 \in \gamma(0)$ and $g(0) = 0$, that $(-\Delta u, A_\lambda u) \ge 0$ for $u \in D(\partial\varphi)$ and $(-\Delta u, A_H u) \ge 0$ for $u \in D(\partial\varphi)$ (see [6, Cor. 13]). Thus all the assumptions of Theorem 1 hold with these choices of φ, ψ, W, H provided a satisfies Conditions a. We conclude that if $t \to F(t,\cdot) \in W^{1,1}_{loc}([0,\infty), L^2(\Omega))$ and $u_0 \in D(\varphi)$ (6.1),

(6.2), (6.3) has a solution $u(t,x)$, $t \geq 0$, $x \in \Omega$ such that $t \to u(t,\cdot)$ satisfies the conclusions of Theorem 1 with the current choices of φ, ψ, W, H. We also have $\varphi, \psi \geq 0$ here, so in view of the remarks following Theorem 3 if, e.g., $F \equiv 0$ and a satisfies Conditions (a_2), then $\int_\Omega G(u(t,x))dx$ is bounded. To have $\lim_{\|u\| \to \infty} \psi(u) = \infty$ we require $|g(u)| \geq c_2(|u|^{p-1} - 1)$ for some $c_2 > 0$. Then (1.15) holds. If also $a'(t) \not\equiv 0$, then $t \to g(u(t,\cdot))$ tends to zero weakly in $L^{p/(p-1)}(\Omega)$ as $t \to \infty$. If also $g^{-1}(0) = \{0\}$, then $A^{-1}(0) = \{0\}$ and by (1.17) $u(t,x) \to 0$ in $L^p(\Omega)$ as $t \to \infty$.

The analogue of (6.1) - (6.3) with infinite delay is

$u_t - \Delta u + \int_{-\infty}^{t} a(t-s)g(u(s))ds = f(t)$, (6.2) and $u(t) = h(t)$ $(-\infty < t \leq 0)$,

where we surpress the dependence on x temporarily. *According to* Proposition 5.1, if $h(0) \in D(\varphi)$ and

$$\int_{-\infty}^{0} \left(\int_\Omega |g(h(t,x))|^2 dx\right)^{1/2} dt < \infty$$

we have the existence of a solution of this problem via Theorem 1. If also $f \equiv 0$, a satisfies conditions (a_2), $|g(r)| \geq c_2(|r|^{p-1} - 1)$ and $g^{-1}(0) = \{0\}$, we conclude $u(t,\cdot) \to 0$ in $L^p(\Omega)$ by arguing as above and using Proposition 5.2.

In comparison with [2], we have eliminated the condition $g(u) \geq c_2(|u|^{p-1} - 1)$ as a hypothesis for existence of solutions of (6.1) - (6.3) and are able to make assertions concerning the asymptotic behavior of the solutions without restriction on N (and other restrictive

conditions on a used in [2]). Moreover, the case with infinite delay is accommodated without further ado.

Example 2. To illustrate Theorem 2, consider the problem

$$(6.9) \qquad u_t(t,x) - u_{xx}(t,x) - \int_0^t a(t-s)(\sigma(u_x(s,x)))_x ds = F(t,x)$$

for $t > 0$, $0 < x < 1$ with the boundary and initial conditions

$$(6.10) \qquad u(t,0) = u(t,1) = 0, \ t > 0 \ \text{ and } \ u(0,x) = u_0(x), \quad 0 \le x \le 1 .$$

Assume the nonlinear function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies

$$(6.11) \qquad \sigma \in C^1(-\infty,\infty), \ 0 \le \sigma' \le M < \infty$$

and

$$(6.12) \qquad \Sigma(r) = \int_0^r \sigma(s)ds \ge c(r^2 - 1) \quad \text{for some} \quad c > 0 .$$

Let $W = H = L^2(0,1)$ and $\psi : L^2(0,1) \to (-\infty,\infty]$ be defined by

$$(6.13) \qquad \psi(u) = \begin{cases} \int_0^1 \Sigma(\frac{du}{dx})dx & \text{if } u \in H_0^1(0,1) \\ +\infty & \text{otherwise} . \end{cases}$$

ψ is well-defined, proper and convex by (6.11), (6.12) and l.s.c. by (6.12). Moreover

$$(6.14) \quad \partial\psi(u) = -\frac{d}{dx}(\sigma(\frac{du}{dx})), \ u \in D(\partial\psi) = \{u \in H_0^1 : \frac{d}{dx}\sigma(\frac{du}{dx}) \in L^2(0,1)\} .$$

It is easy to see that $\partial\psi$ is an extension of the operator given by (6.14). To see the equality, show (6.14) is the restriction of the subdifferential of ψ regarded as a map from $H_0^1(0,1)$ to $\mathbb{R}$ to H (which is easy to

compute), so (6.14) is maximal monotone. We let $\varphi : L^2(0,1) \to (-\infty, \infty]$ be given by

$$(6.15) \qquad \varphi(u) = \begin{cases} \frac{1}{2} \int_0^1 \left|\frac{du}{dx}\right|^2 dx, & u \in H_0^1(0,1) \\ +\infty & \text{otherwise} . \end{cases}$$

Clearly $\{u : |\varphi(u)| + |u| \le K\}$ is compact in $L^2(0,1)$ for each K. Moreover, if $u \in D(\partial\varphi)$ (6.11) implies

$$|\partial\varphi(u)|^2 = \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx \ge \frac{1}{M^2} \int_0^1 \left(\left(\sigma'\left(\frac{du}{dx}\right)\right)^2 \left(\frac{d^2u}{dx^2}\right)\right)^2 dx = \frac{1}{M^2} |Au|^2$$

and (1.12) holds.

The key hypotheses to verify is (1.7). This does not seem immediate to us. Let $u \in D(\partial\varphi)$, $h \in H_0^1(0,1)$, $\lambda > 0$ and

$$(6.16) \qquad h - \lambda \frac{d}{dx}\left(\sigma\left(\frac{dh}{dx}\right)\right) = u .$$

That is, $h = J_\lambda(u)$ and $-\frac{d}{dx}\sigma\left(\frac{dh}{dx}\right) = A_\lambda(u)$. We will show

$$(6.17) \qquad \varphi(J_\lambda u) = \varphi(h) \le \varphi(u)$$

which implies (1.7) with $\beta = 0$. Now by (6.16)

$$(6.18) \qquad 2\varphi(u) = \int_0^1 \left(\frac{du}{dx}\right)^2 dx = \int_0^1 \left(\frac{d}{dx}\left(h - \lambda \frac{d}{dx}\sigma\left(\frac{dh}{dx}\right)\right)\right)^2 dx$$

$$= \int_0^1 \left(\frac{dh}{dx}\right)^2 + \lambda^2 \int_0^1 \left(\frac{d^2}{dx^2}\sigma\left(\frac{dh}{dx}\right)\right)^2 dx$$

$$-2\lambda \int_0^1 \frac{dh}{dx} \frac{d^2}{dx^2}\sigma\left(\frac{dh}{dx}\right) dx .$$

Note that the indicated derivatives have a meaning by (6.16), since $h, u \in H_0^1(0,1)$. If we knew $h \in H^2(0,1)$, then

$$-2\lambda \int_0^1 \frac{dh}{dx} \frac{d^2}{dx^2} \sigma\left(\frac{dh}{dx}\right) dx = 2\lambda \int_0^1 \frac{d^2h}{dx^2} \frac{d}{dx} \sigma\left(\frac{dh}{dx}\right) dx$$

$$= 2\lambda \int_0^1 \sigma'\left(\frac{dh}{dx}\right)\left(\frac{d^2h}{dx^2}\right)^2 dx \geq 0 .$$

Hence all terms on the right of (6.18) are nonnegative and $\varphi(u) \geq \varphi(h)$ as desired. If $\sigma' \geq \varepsilon > 0$ for some ε, (6.16) implies $h \in H^2(0,1)$ since $\frac{dh_\varepsilon}{dx} = \sigma^{-1}\left(\sigma\left(\frac{dh_\varepsilon}{dx}\right)\right)$ and $\sigma\left(\frac{dh_\varepsilon}{dx}\right) \in H^1(0,1)$. To proceed, let $h_\varepsilon \in H_0^1(0,1)$ satisfy

$$h_\varepsilon - \lambda \frac{d}{dx}\left(\sigma\left(\frac{dh_\varepsilon}{dx}\right) + \varepsilon \frac{dh_\varepsilon}{dx}\right) = u \tag{6.19}$$

where $\varepsilon > 0$. Since $r \to \sigma_\varepsilon(r) = \sigma(r) + \varepsilon r$ satisfies $\sigma_\varepsilon' \geq \varepsilon$, $\varphi(h_\varepsilon) \leq \varphi(u)$ by the above. Multiplying (6.19) first by h_ε and integrating we find

$$\int_0^1 h_\varepsilon^2 dx \leq \int_0^1 \left(h_\varepsilon^2 + \lambda\sigma\left(\frac{dh_\varepsilon}{dx}\right)\frac{dh_\varepsilon}{dx} + \varepsilon\left(\frac{dh_\varepsilon}{dx}\right)^2\right) dx$$

$$= \int_0^1 \left(h_\varepsilon - \lambda \frac{d}{dx}\left(\sigma\left(\frac{dh_\varepsilon}{dx}\right) + \varepsilon \frac{dh_\varepsilon}{dx}\right)\right) h_\varepsilon dx$$

$$= \int_0^1 u h_\varepsilon dx ,$$

so

$$\|h_\varepsilon\|_{L^2(0,1)} \leq \|u\|_{L^2(0,1)} . \tag{6.20}$$

Next multiply (6.19) by $\left(-\dfrac{d^2h_\varepsilon}{dx^2}\right)$ and integrate to find

$$\int_0^1 \left(\frac{dh_\varepsilon}{dx}\right)^2 dx + \lambda \int_0^1 \left(\sigma'\left(\frac{dh_\varepsilon}{dx}\right)\left(\frac{d^2h_\varepsilon}{dx^2}\right)^2 + \varepsilon\left(\frac{d^2h_\varepsilon}{dx^2}\right)^2\right)dx$$

$$= -\int_0^1 u \frac{d^2h_\varepsilon}{dx^2} dx = -\int_0^1 \frac{d^2u}{dx^2} h_\varepsilon dx$$

$$\leq \left\|\frac{d^2u}{dx^2}\right\|_{L^2(0,1)} \|u\|_{L^2(0,1)}$$

where (6.20) is used in the last inequality. Thus

$$\varepsilon \int_0^1 \left(\frac{d^2h_\varepsilon}{dx^2}\right)^2 dx \leq C$$

where C is independent of ε and therefore $\varepsilon \dfrac{d^2h_\varepsilon}{dx^2} \to 0$ in $L^2(0,1)$ as $\varepsilon \downarrow 0$. Since $h_\varepsilon = J_\lambda\left(u - \lambda\varepsilon \dfrac{d^2h_\varepsilon}{dx^2}\right)$, and J_λ is a contraction, $h_\varepsilon \to h$ in $L^2(0,1)$ as $\varepsilon \downarrow 0$. Since φ is l.s.c.,

$$\varphi(h) \leq \liminf_{\varepsilon \downarrow 0} \varphi(h_\varepsilon) \leq \varphi(u)$$

and we are done. We are grateful to L. Tartar for an earlier proof of the above result.

Thus, according to Theorem 2, if $t \to F(t,\cdot)$ is in $W^{1,1}_{loc}(0,\infty; L^2(0,1))$ and $u_0 \in H^1_0(0,1)$ then (6.9)-(6.10) has a solution $u(t,x)$ with $t \to u(t,\cdot)$, $u_t(t,\cdot)$, $u_{xx}(t,\cdot)$ and $\sigma(u_x(t,\cdot))_x$ all in $L^2_{loc}(0,\infty; L^2(0,1))$.

We can illustrate Theorem 4 here as well. Assume that $t \to F(t,\cdot)$ also lies in $L^2(0,\infty; L^2(0,1))$. Since by (6.10)

$$(Bu, Au) = \int_0^1 \frac{d^2u}{dx^2}\, \sigma'\left(\frac{du}{dx}\right)\frac{d^2u}{dx^2}\, dx \geq \frac{1}{M} |Au|^2$$

we have (1.18) (i) with $\alpha = 1/M$. Hence if

$$\delta = \alpha + \limsup_{T\to\infty} \inf_{-\infty<\sigma<\infty} \int_0^T \cos(\sigma t)a(t)dt > 0 \ ,$$

(6.12) and Theorem 4 imply

$$c \int_0^1 (u_x(t,x))^2 dx \leq c + \sup_{t>0} \psi(u(t,\cdot)) < \infty \ ,$$

and

$$\int_0^\infty \int_0^1 \left|\frac{\partial}{\partial x}\, \sigma(u_x(t,x))\right|^2 dx dt < \infty \ .$$

Remark. *Barbu [2] mentions (6.9), (6.10) under his further restrictions* on a but with a weaker assumption than σ' is bounded above. He does not verify (1.7).

Appendix (a)

Theorem (a). Let $a : [0,\infty) \to \mathbb{R}$ satisfy the following conditions:

$$(1)\quad \begin{cases} a \ \text{is locally absolutely continuous on}\ [0,\infty)\ \text{and} \\ a' \ \text{is locally of bounded variation on}\ (0,\infty)\,. \end{cases}$$

$$(2)\quad \begin{cases} \text{There are constants}\ \ell, T > 0\ \text{such that} \\ \int_0^T \operatorname{var}(a', [s, s+\ell])ds < \infty \ \text{where}\ \operatorname{var}(a', I)\ \text{is} \\ \text{the total variation of}\ a'\ \text{over}\ I\,. \end{cases}$$

$$(3)\quad \begin{cases} \text{There are constants}\ \gamma, T_0 > 0,\ \gamma > \eta\ \text{such that} \\ Q_a(v;t) = \int_0^t (a * v(s), v(s))ds \\ \qquad \geq \gamma \left| \int_0^t v(s)ds \right|^2 - \eta \max\limits_{0 \leq \tau \leq t} \left| \int_0^\tau v(s)ds \right|^2 \\ \text{for}\ 0 \leq t \leq T_0\ \text{and every}\ v \in L^2(0, T_0; H)\,. \end{cases}$$

Then a satisfies Conditions (a).

We preceed the proof of Theorem (a) with the proof of Proposition (a) (which is stated in the introduction).

Proof of Proposition (a). Consider at first the case when conditions (a_1) are satisfied. It follows trivially that then (1), (2) hold for any $\ell, T > 0$. To obtain (3) we begin by using the identity (for a proof of (4) under conditions (a_1) see e.g. [11]),

$$(4)\quad \begin{cases} Q_a(v;t) = \dfrac{a(t)}{2} \left| \int_0^t v(\tau)d\tau \right|^2 - \dfrac{1}{2} \int_0^t a'(\tau) \left| \int_0^\tau v(s)ds \right|^2 d\tau \\ \quad - \dfrac{1}{2} \int_0^t a'(t-\tau) \left| \int_\tau^t v(s)ds \right|^2 d\tau + \dfrac{1}{2} \int_0^t \int_0^\tau \left| \int_{\tau-s}^\tau v(u)du \right|^2 da'(s)d\tau\,, \end{cases}$$

where $v \in L^2_{loc}(0,\infty; H)$. Then we notice that simple estimates on the right side of (4) give

$$Q_a(v;t) \geq \frac{a(t)}{2} \left|\int_0^t v(\tau)d\tau\right|^2 - b(t) \sup_{0 \leq \tau \leq t} \left|\int_0^\tau v(s)ds\right|^2$$

where $b(t) = \frac{5}{2}\int_0^t |a'(s)|ds + 2\int_0^t \int_0^\tau |da'(s)|d\tau$. Choosing $T_0 > 0$ such that $4b(T_0) \leq \inf_{0 \leq t \leq T_0} a(t)$ shows that (3) holds with this T_0, $\gamma = \inf_{0 \leq t \leq T_0} \frac{a(t)}{2}$, and $2\eta = \gamma$. Thus conditions (a_1) imply (1) - (3).

Next let conditions (a_2) hold. Observe that this case does allow $a'(0+) = -\infty$. As in the previous case it immediately follows that (1), (2) are valid for any $\ell, T > 0$. (To obtain (2) use the monotonicity of a'). Then notice that a simple application of the dominated convergence theorem shows that (4) holds under conditions (a_2), for $v \in L^2_{loc}(0,\infty; H)$. But by (a_2) all the terms on the right side of (4) are nonnegative and so (3) holds with any $T_0 > 0$ such that $a(T_0) > 0$, with $2\gamma = a(T_0)$ and $\eta = 0$. Hence conditions (a_2) imply (1) - (3).

Proof of Theorem (a). First notice that it is enough to show that if c_1, c_2, T are arbitrary nonnegative numbers then there is a constant $c_3 = c_3(a, c_1, c_2, T)$ such that for every $v \in L^2(0, T; H)$ satisfying the inequality

$$(5) \qquad Q_a(v;t) \leq c_1 + c_2 \max_{0 \leq \tau \leq t} \left|\int_0^\tau v(s)ds\right|, \quad (0 \leq t \leq T)$$

one has

$$(6) \qquad \left|\int_0^t v(s)ds\right| \le c_3, \quad |Q_a(v;t)| \le c_3, \quad (0 \le t \le T) .$$

For then (5) implies $w = \dfrac{1}{\sqrt{c_1} + c_2} v$ satisfies

$$Q_a(w;t) = \left(\frac{1}{\sqrt{c_1} + c_2}\right)^2 Q_a(v;t) \le 1 + \max_{0 \le \tau \le t} \left|\int_0^\tau w(s)ds\right|$$

and so

$$\left|\int_0^t v(s)ds\right| = (\sqrt{c_1} + c_2)\left|\int_0^t w(s)ds\right| \le c_3(a,1,1,T)(\sqrt{c_1} + c_2) .$$

Similarly, one estimates $Q_a(v;t)$ and finds that $K_T = 2c_3(a,1,1,T)$ works in Conditions (a).

Let $c_1, c_2 > 0$ be arbitrary and (5) hold. It clearly suffices to consider $T = nT_0$ where T_0 is as in (3) and n is an arbitrary positive integer. The proof is by induction on n.

By (3) and (5) one easily obtains the existence of a constant $c_3 > 0$ such that the estimates (6) hold if $T = T_0$. Assume we can find such a constant for $T = nT_0$ (denote this constant by K_1) and let $v \in L^2(0,(n+1)T_0; H)$ satisfy (5) for $0 \le t \le (n+1)T_0$. Thus

$$(7) \qquad \left|\int_0^t v(s)ds\right| \le K_1, \quad |Q_a(v;t)| \le K_1, \quad (0 \le t \le nT_0) .$$

For $t \in [0, T_0]$ one obviously has

$$(*) \qquad Q_a(v;t + nT_0) = Q_a(v;nT_0) + \int_0^t (a * v(s + nT_0), v(s + nT_0))ds .$$

Also note that $(a * v)(s + nT_0) = I_1 + I_2$, where

$$I_1 = \int_0^{nT_0} a(s + nT_0 - \xi)v(\xi)d\xi, \; I_2 = \int_0^{s} a(s - \xi)v(\xi + nT_0)d\xi \; .$$

Substituting these relations into the integrand of the last term of (*) and writing $v_{nT_0}(s) = v(s + nT_0)$ gives

$$(8) \quad \begin{cases} Q_a(v;t + nT_0) = Q_a(v;nT_0) + Q_a(v_{nT_0};t) \\ \quad + \int_0^t (\int_0^{nT_0} a(s + nT_0 - \xi)v(\xi)d\xi, \; v_{nT_0}(s))ds, \quad (0 \le t \le T_0) \end{cases}$$

where the last term comes from I_1 and the second term on the right of (8) comes from I_2.

Suppose we can show that there exist constants M_1, m_1, independent of v, such that

$$(9) \qquad |J(t)| \le M_1 + m_1 \max_{0 \le \tau \le t} |\int_0^{\tau} v_{nT_0}(s)ds| \quad (0 \le t \le T_0)$$

where $J(t) = \int_0^t (\int_0^{nT_0} a(s + nT_0 - \xi)v(\xi)d\xi, \; v_{nT_0}(s))ds$. Then (5) with $T = (n+1)T_0$, (7) and (9) used in (8) imply the existence of constants M_2, m_2, independent of v, such that

$$Q_a(v_{nT_0};t) \le M_2 + m_2 \max_{0 \le \tau \le t} |\int_0^{\tau} v_{nT_0}(s)ds| \quad (0 \le t \le T_0) \; .$$

Consequently by the case $n = 1$, already proved, we have the existence of a constant $K_2 > 0$ such that

$$(10) \qquad |\int_0^t v_{nT_0}(s)ds| \le K_2, \; |Q_a(v_{nT_0};t)| \le K_2, \quad (0 \le t \le T_0) \; .$$

But then (7), (9), and (10) used in (8) give the existence of a constant $K_3 > 0$ such that

$$|Q_a(v;t + nT_0)| \leq K_3, \quad (0 \leq t \leq T_0) .$$

Moreover, from the first parts of (7) and (10) one has

$$\left| \int_0^{t+nT_0} v(s)ds \right| \leq \left| \int_0^{nT_0} v(s)ds \right| + \left| \int_0^{t} v_{nT_0}(s)ds \right| \leq K_1 + K_2, \quad (0 \leq t \leq T_0) .$$

The induction argument is hence complete provided we can establish (9).

To prove (9) we proceed as follows. Integrating the expression for $J(t)$ by parts (justified by conditions (1), (2)) we have $J = J_1 + J_2$ where

$$J_1(t) = \left(\int_0^{nT_0} a(t + nT_0 - \xi)v(\xi)d\xi, \int_0^{t} v_{nT_0}(s)ds \right) \quad (0 \leq t \leq T_0) ,$$

$$J_2(t) = -\int_0^{t} \left(\frac{d}{ds} \int_0^{nT_0} a(s + nT_0 - \xi)v(\xi)d\xi, \int_0^{s} v_{nT_0}(\xi)d\xi \right) ds, \quad (0 \leq t \leq T_0) .$$

Integrating the first factor in the expression for J_1 by parts gives

$$\int_0^{nT_0} a(t + nT_0 - \xi)v(\xi)d\xi = a(t) \int_0^{nT_0} v(\xi)d\xi + \int_0^{nT_0} a'(t + nT_0 - \xi) \int_0^{\xi} v(\tau)d\tau d\xi$$

which, when used in the expression for $J_1(t)$ (also apply the first part of (7) and make obvious estimates) gives

$$(**) \quad |J_1(t)| \leq K_1 \left[|a(t)| + \int_0^{t+nT_0} |a'(s)|ds \right] \int_0^{t} |v_{nT_0}(s)|ds, \quad (0 \leq t \leq T_0) .$$

Estimating the expression for J_2 one obtains

$$|J_2(t)| \le \{\int_0^t |\frac{d}{ds}\int_0^{nT_0} a(s+nT_0-\xi)v(\xi)d\xi|ds\}\{\max_{0\le\tau\le t} |\int_0^\tau v_{nT_0}(s)ds|\}$$

$$= \{\int_0^t |\frac{d}{ds}[a(s)\int_0^{nT_0} v(\xi)d\xi + \int_0^{nT_0} a'(s+nT_0-\xi)\int_0^\xi v(\tau)d\tau d\xi]|ds\} \times$$

$$\{\max_{0\le\tau\le t} |\int_0^\tau v_{nT_0}(s)ds|\} \quad (0 \le t \le T_0),$$

where the equality follows after an integration by parts. By conditions (1), (2) the quantities $\int_0^{nT_0} a(s+nT_0-\xi)v(\xi)d\xi$ and $a(s)\int_0^{nT_0} v(\xi)d\xi$ have derivatives in $L^1(0,T_0;H)$; therefore the same applies to the derivative of $\int_0^{nT_0} a'(s+nT_0-\xi)\int_0^\xi v(\tau)d\tau d\xi$. But the L^1-norm of the derivative equals the total variation. Hence, denoting an arbitrary partition of $[0,T_0]$ by $0 = s_0 < s_1 \ldots < s_N = T_0$, it follows that

$$\sum_{i=1}^N |\int_0^{nT_0} a'(s_i+nT_0-\xi)\int_0^\xi v(\tau)d\tau d\xi - \int_0^{nT_0} a'(s_{i-1}+nT_0-\xi)\int_0^\xi v(\tau)d\tau d\xi|$$

$$\le \sum_{i=1}^N \int_0^{nT_0} |a'(s_i+nT_0-\xi) - a'(s_{i-1}+nT_0-\xi)|d\xi \max_{0\le\tau\le nT_0} |\int_0^\tau v(s)ds|$$

$$\le K_1 \int_0^{nT_0} \mathrm{Var}(a',[nT_0-\xi,(n+1)T_0-\xi])d\xi = K_4 < \infty,$$

where the last steps follow by (2) and (7). Thus

$$|J_2(t)| \le K_4 \max_{0\le\tau\le t} |\int_0^\tau v_{nT_0}(s)ds| .$$

Combining this relation with (**) and recalling that $J = J_1 + J_2$ implies (9) and completes the proof of Theorem (a).

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We study the nonlinear Volterra equation was studied

$$(*)\quad \begin{cases} u'(t) + Bu(t) + \int_0^t a(t-s)Au(s)ds \ni F(t) & (0 < t < \infty)\ \ (' = d/dt) \\ u(0) = u_0, & \end{cases}$$

as well as the corresponding problem with infinite delay. → (continued) over

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$$(**)\quad \begin{cases} u'(t) + Bu(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds \ni f(t) & (0 < t < \infty) \\ u(t) = h(t) & (-\infty < t \leq 0)\ . \end{cases}$$

Under various assumptions on the nonlinear operators A, B and on the given functions a, F, f, h **existence** theorems are obtained for (*) and (**), followed by results concerning boundedness and asymptotic behaviour of solutions on $(0 \leq t < \infty)$; two applications of the theory to problems of nonlinear heat flow with "infinite memory" are also discussed.

LESS THAN OR EQUAL TO

INFINITY

LESS THAN

NONLINEAR VOLTERRA EQUATION

NONLINEAR VOLTERRA EQUATION
WITH INFINITE DELAY